

Some bounds upon nonextensivity parameter using the approximate generalized distribution functions

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In this study, approximate generalized quantal distribution functions and their applications, which appeared in the literature so far, have been summarized. Making use of the generalized Planck radiation law, which has been obtained by the authors of the present manuscript [Physica A240 (1997) 657], some alternative bounds for nonextensivity parameter q has been estimated. It has been shown that these results are similar to those obtained by Tsallis et al. [Phys.Rev. E52 (1995) 1447] and by Plastino et al. [Phys.Lett. A207 (1995) 42].

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I. INTRODUCTION

Boltzmann-Gibbs (BG) statistics provides a very powerful and suitable tool for handling physical systems where thermodynamic extensivity (additivity) holds. Apparently, this condition itself points out that BG formalism has some restrictions. These could be stated as follows: (i) the spatial ranges of the microscopic interactions must be short, (ii) the time range of the microscopic memory must be short and (iii) the system must evolve in an Euclidean-like, nonfractal, space-time. Whenever one and/or the others of these restrictions are violated, BG formalism fails to study the system under consideration. This kind of systems are said to be nonextensive, therefore a nonextensive formalism of statistics is needed for them. In the recent years, an increasing tendency towards nonextensive formalisms keeps growing along two lines : Quantum-Group-like approaches [1] and Tsallis Thermostatistics (TT) [2], to which this paper is dedicated.

TT, which has been proposed by C. Tsallis in 1988, is based upon two basic axioms:

(a) the entropy is given by

$$S_q = -k \frac{1 - \sum_{i=1}^W p_i^q}{1 - q} \quad (1)$$

where k is a positive constant, p_i is the probability of the system to be in a microstate, W is the total number of configurations and q is a real parameter.

(b) the q -expectation value of an observable O is given by

$$\langle O \rangle_q = \sum_{i=1}^W p_i^q O_i \quad (2)$$

It is worthwhile to note that TT contains BG statistics as a special case in the $q \rightarrow 1$ limit. Extensive Shannon entropy and standard definition of the expectation value are recovered in this limit, thus $(1 - q)$ could be interpreted as the measure of the lack of extensivity of the system.

In fact, it is plausible to classify the works on this subject in the following manner: (i) Some of the works have been devoted to the generalization of the conventional concepts of thermostatics so far, in order to see whether this nonextensive formalism allows an appropriate extension of these concepts. (ii) After these

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efforts, TT has been applied to some physical systems [3] where BG formalism is known to fail, and its success on these lines accelerates new attempts nowadays. (iii) Finally, the last area of interest, which is very active, originates from the fact that the physical meaning of the q -index has only now started to be clarified. Efforts on this field accumulate along two main streams; one of which is the study of dynamical systems [4] in order to find out the connection between the q -index and the dynamical parameters of the system, whereas the second one is the attempts of establishing some bounds upon q by studying, in general, astrophysical problems such as the cosmic microwave background radiation, Planck law [5,6,7] and the early Universe [8,9]. In this study, our main goal is to contribute to this stream by estimating some alternative bounds upon q .

Before closing this Section, it could be mentioned that the reader can find a detailed review of TT with its important properties in [10], moreover, a full bibliography of TT is now available in [11].

II. GENERALIZED QUANTAL DISTRIBUTION FUNCTIONS

Amongst the generalizations of the concepts of thermostatics in the frame of TT, quantal distribution functions have been studied for the first time by our group [12,13]. In [12], using the statistical weights, the fractal inspired entropies of quantum gases have been deduced and the generalized distribution functions have been found by introducing the constraints related to the statistical properties of the particles. In [13], Tsallis entropy has been used for the same purpose and the same results for the generalized distribution functions have been reproduced but in this case the constraints have been imposed in the calculations of the partition functions.

Very recently, these distribution functions have been reobtained by Kaniadakis and Quarati [14], using a completely different technique. Since the classical statistical distributions could be viewed as stationary states of a linear Fokker-Planck equation, analogously, the authors have shown that [15] quantal distributions can be obtained, by means of a semiclassical approach, as stationary states of a nonlinear Fokker-Planck equation. They have found that; within this kinetic approach, a family of classical and quantal distributions can be derived when the drift and the diffusion coefficients are polynomials of $2M + 1$ and $2N$ degrees, respectively, in the velocity variable. A particular and well-defined distribution corresponds to each pair of (M, N) , i.e. $(0, 1)$ gives the generalized distribution functions which are identical with the results of [12,13]. It is also worth mentioning that some aspects of generalized quantum statistics can be found in [16].

The generalized quantal distribution functions, derived independently in these three papers, are given by

$$\langle n_r \rangle_q = \frac{1}{[1 - (1 - q)\beta(\epsilon_r - \mu)]^{1/(q-1)} + \kappa} \quad (3)$$

where κ is $+1$ and -1 for Fermi-Dirac and Bose-Einstein distributions, respectively, μ being the chemical potential.

It is worth emphasizing that these expressions for the quantal distribution functions are not exact results due to the approximate scheme used in the calculations. This scheme is generically based on the factorization of the generalized grand canonical distribution function and the concomitant q -partition function as if they were extensive quantities, due to the fact that a closed, manageable form for them cannot be found without making such approximations. The authors of the present manuscript have recently discussed the nature of this approximation in [7] more clearly and showed that these results give upper or lower bounds to the exact results with respect to the values of q -index. The role of this approximation is very similar to the role of the mean-field theory approximation used in the theory of phase transitions. This approximate scheme has also been used in another problem [17] and it is seen that it gives similar results with the prior works on the same subject.

III. APPLICATIONS OF THE GENERALIZED QUANTAL DISTRIBUTION FUNCTIONS

Although the generalized distribution functions are first introduced in 1993, there has been no attempt to apply them to the physical systems until a recent effort performed by Pennini et al. [18]. The authors dealt with N independent particles (fermions and bosons), distributed among M single particle levels of single particle energies ϵ_i ($i = 1, \dots, M$). In order to obtain the exact result, the authors maximized the Tsallis entropy with suitable constraints and found the exact result of an extremely simple model, which is the simplest conceivable situation for fermions where $M = 2$, since otherwise the non-separability of

the partition function does not allow to find a close, analytical expression for arbitrary q values without making an approximation like the one we discussed above. The average fermion occupation number of the lowest single particle level versus inverse temperature with $q = 0.7$ is given in Fig. 1 of [18]. Pennini et al. performed the same calculations for bosons as well with $M = 2$. The average boson occupation number of the lowest single particle level with $q = 0.7$ is given in Fig. 3 of [18]. In both figures, it is clearly seen that our approximate scheme provides bounds to the exact results (since Pennini et al. used $q < 1$, the bound appears as a lower bound). This study also suggests that the quality of our approximation deteriorates as the number of particles of the system increases, however, a very recent effort by Wang and Le Méhauté [19] showed clearly that there exist a temperature interval where the ignorance of the approximation is significant, but outside this interval, the results of this approach can be used with confidence. In addition, they verified that the magnitude of this interval remains *constant* with the increase of the number of particles, a rather nice result in the favour of the approximate scheme, contrary to the result of Pennini et al.

The second application of the generalized distribution functions has recently been performed by the authors of this manuscript [7]. Two years ago, the generalization of the Planck law was obtained by Tsallis et al. [5] for $q \cong 1$, in order to see whether present cosmic background radiation is (slightly) different from the Planck radiation law due to the long-range gravitational influence. We have advanced an alternative approach to this generalization by using one of the generalized distribution functions, namely generalized Planck distribution. Once again, as expected, our approximate scheme shows itself as an upper or lower bound to the exact result, depending on the value of q . This fact is verified clearly in the figure of [7] by plotting the blackbody photon energy density per unit volume versus $h\nu/kT$ in the frame of [7] and [5]. We have also been able to obtain all the generalized laws, which was found by Tsallis et al., (i.e., generalized Planck, Rayleigh-Jeans, Stefan-Boltzmann and Wien laws) and noticed that our approach is simpler than that of Tsallis et al. since the procedure we have used is completely the same as that followed in any standard textbook of Statistical Physics for the derivation of the standard Planck law and also our approximation which leads bounds seems to be more general, since not necessarily $q \cong 1$.

IV. ESTIMATION OF SOME BOUNDS UPON Q

In [7], the generalized Planck radiation law is given by

$$D_q(\nu) \simeq \frac{8\pi k^3 T^3}{c^3 h^2} \frac{x^3}{[1 - (1 - q)x]^{\frac{1}{q-1}} - 1} \quad (4)$$

where $x \equiv h\nu/kT$. For our purpose, it is straightforward to expand this expression to first order in $(1 - q)$, which reads

$$D_q(\nu) \simeq \frac{8\pi k^3 T^3}{c^3 h^2} \left[\frac{x^3}{e^x - 1} - \frac{1}{2}(1 - q) \frac{x^5 e^x}{(e^x - 1)^2} \right]. \quad (5)$$

For a comparison, eq(5), together with the results of [7] and [5] for $D_q(\nu)$, have been illustrated in the Figure.

The value of the frequency ν_m , which makes $D_q(\nu)$ maximum, can easily be found by maximizing eq.(5):

$$x_m \equiv h\nu_m/kT \simeq a_1 + a_2(q - 1) \quad (6)$$

where $a_1 \simeq 2.821$ and $a_2 \simeq 4.928$. Substitution of this result in eq.(5) immediately yields

$$D_q(\nu_m) \frac{c^3 h^2}{8\pi k^3 T^3} \simeq d_1 + d_2(q - 1) \quad (7)$$

with $d_1 \simeq 1.421$ and $d_2 \simeq 10.677$. Lastly, we verify that, for $\nu \simeq \nu_m$,

$$\frac{D_q(\nu)}{D_q(\nu_m)} \simeq 1 - B \left(\frac{\nu}{\nu_m} - 1 \right)^2 \quad (8)$$

where

$$B \simeq b_1 - b_2(q - 1) \quad (9)$$

with $b_1 \simeq 1.232$ and $b_2 \simeq 2.322$. From these results, the following expression can easily be written:

$$q - 1 \simeq \frac{b_1 - B^{exp}}{b_2} \quad (10)$$

which is nothing but an alternative estimation of a bound upon q , similar to that of given in [5] by Tsallis et al. (here, B^{exp} stands for the experimental value of B). Note that although, at first sight, eq.(10) seems to be the same with eq.(24) of [5], in fact they are different since the coefficient b_2 is not equal to the one given in [5]. Strickly speaking, it must be pointed out that all of the coefficients which belong to the nonextensivity part (namely, a_2 , b_2 , d_2) differ from those given in [5].

In the remainder of this manuscript, we focus our attention to another bound upon q , which was established by Plastino et al. [6] with the help of the expression of $D_q(\nu)$ given in [5] and the experimental value of the Stefan-Boltzmann constant. Following the same procedure in [6], but using eq.(5) for $D_q(\nu)$, we establish another bound upon q .

Let us write down the total emitted power per unit surface,

$$P_q = \int_0^\infty D_q(\nu) d\nu = \sigma_q T^4 \quad (11)$$

where σ_q is the generalized Stefan-Boltzmann constant, which is given by

$$\sigma_q \cong \frac{8\pi k^4}{c^3 h^3} \int_0^\infty \left[\frac{x^3}{e^x - 1} - \frac{1}{2}(1 - q) \frac{x^5 e^x}{(e^x - 1)^2} \right] dx \quad (12)$$

If we denote the integral appearing in the above expression by I , the explicit numerical solution reads

$$I = 6.4939 - 62.2157(1 - q) \quad (13)$$

therefore,

$$\sigma_q = 5.67 \times 10^{-8} - 3.49 \times 10^{-7}(1 - q) \quad (14)$$

Using this, together with

$$\sigma_{Planck} = 5.67051(19) \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-2} \quad (15)$$

and the '3 sigma' rule [6]

$$\left| \frac{[(\delta\sigma_{Planck})^2 + (\delta\sigma_q)^2]^{1/2}}{\sigma_q} \right| \leq 4.1 \times 10^{-4} \quad (16)$$

it is possible to find a bound upon q :

$$|q - 1| \leq 0.41 \times 10^{-4} \quad (17)$$

which is similar to that obtained by Plastino et al. [6].

V. CONCLUSIONS

In this study, we have concentrated on estimating new bounds upon nonextensivity parameter q . For this purpose, the generalized Planck radiation law has been used and following the same procedure of [5] and [6], two new bounds for q has been estimated. These results are, as expected, consistent with the previous ones, therefore the same conclusions are valid for the present case.

On the other hand, very recently we have realized that the present effort might motivate some other works on the early Universe. More precisely, Torres et al. [8], within an early Universe scenario, calculated the deviation in the primordial Helium abundance and obtained a bound for q . Moreover, Torres and Vucetich studied the primordial neutron to baryon ratio in a cosmological expanding background [9] and within this context, they found another bound upon q . In both works, the generalized mean value of the particle number operator $\langle n \rangle_q$ has been used. Hence, it is clear that these works could be handled again by using the suitable definition of $\langle n \rangle_q$ given in eq.(3).

Summing up, we applied one of the generalized quantal distribution functions to the microwave background radiation in order to estimate new bounds upon q and further attempts on this line would highly be welcomed.

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FIGURE CAPTION

Figure : Blackbody photon energy density per unit volume versus $h\nu/kT$ in the frame of ref.[5], ref.[7] and this work, for $q = 0.95$, $q = 1$ and $q = 1.05$.

Figure

